

# The bipartite rationing problem

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## Abstract

In the *bipartite rationing* problem, a set of agents share a single resource available in different “types”, each agent has a claim over only a subset of the resource-types, and these claims overlap in arbitrary fashion. The goal is to divide fairly the various types of resource between the claimants, when resources are in short supply.

With a single type of resource, this is the standard rationing problem (O’Neill [34]), of which the three benchmark solutions are the proportional, uniform gains, and uniform losses methods. We extend these methods to the bipartite context, imposing the familiar *consistency* requirement: the division is unchanged if we remove an agent (resp. a resource), and take away at the same time his share of the various resources (resp. reduce the claims of the relevant agents). The uniform gains and uniform losses methods have infinitely many consistent extensions, but the proportional method has only one. In contrast, we find that most parametric rationing methods (Young [42], [39]) cannot be consistently extended.

## 1 Introduction

We consider the problem of dividing a max-flow in an arbitrary bipartite graph between source and sink nodes. Each source node holds a finite amount of the commodity (say homogeneous freight; more examples below), each sink has a finite capacity to store it, and all the edges have infinite capacity. If each node wishes to send or receive as much of the commodity as possible<sup>1</sup>, it is optimal to implement a max-flow, but there are typically many of those: our goal is to propose a fair way to select one max-flow in any such problem.

Consider the special case of our problem in which there is a single sink node whose capacity is smaller than the sum of the capacities of all the sources. This is the well-studied problem of rationing a single resource on which the agents

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<sup>1</sup>Alternatively, implementing a maxflow is a design constraint, but each node wishes to process as little of the commodity as possible.

have claims. The simplest rationing method, going back at least to Aristotle, is *proportional*: it divides the resource in proportion to individual claims.<sup>2</sup> To see how the proportional method can be applied in the more general bipartite context, consider the example shown in Figure 1. Two facilities (sinks)  $a, b$

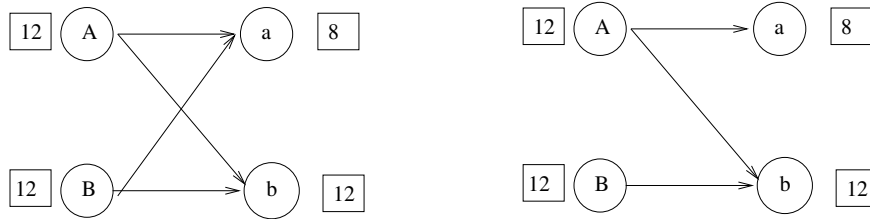


Figure 1: An example with 2 sources and 2 sinks

with capacities 8 and 12 respectively, are shared by two agents (sources) Ann (A) and Bob (B), each requiring 12 units of storage space. The facilities are overdemanded. If both Ann and Bob can ship to both facilities they will share them equally: each of them will ship a total of 10 units (4 units to  $a$  and 6 units to  $b$ ). Now assume Ann can ship to either  $a$  or  $b$ , while Bob can only ship to  $b$  (Figure 1b). The max-flow is still 20, and it is still feasible to let Ann and Bob ship 10 units each, by letting Ann ship only 2 units to  $b$ . Whether or not this is *fair* depends very much on our view of why Bob cannot ship to facility  $a$ .

If Bob’s link to facility  $a$  was destroyed by an “act of God” for which he cannot be held responsible (a storm made the road impassable), compensating for Bob’s handicap by increasing his share of facility  $b$  makes good sense. Not so if Bob’s limitations are of his doing: for example, if he is shipping a perishable commodity that cannot be stored in  $a$ . In the latter case, Ann is entitled to all the capacity of facility  $a$ , which she is the only one to claim. She still competes with Bob for the resources at  $b$ , but her claim on  $b$  cannot be the full 12 units she started with. If it was, and we divided the capacity at  $b$  equally, Ann would end up with 14 units: this is not only unfair but infeasible as well! Clearly Ann only has a *residual* claim of  $12 - 8 = 4$  units on  $b$ , competing with Bob’s claim of 12; Ann’s proportional share of  $b$  is then 3, and she ends up shipping 11 units. The example in Figure 1 illustrates a key principle of our approach: of two agents with identical demands, the one who has a claim on a larger set of the resources should end up with a larger share.

The idea of residual demand, illustrated by the example of Figure 1, is easily generalized to an arbitrary graph. We divide any given facility  $a$  in proportion to the residual claims of all agents  $i$  linked to  $a$ , i.e., agent  $i$ ’s claim on  $a$  is her initial claim *minus* her total allocation of facilities other than  $a$ . This means that our max-flow must solve a certain fixed point system, that we illustrate in the problem depicted by Figure 2. Three agents 1, 2, 3 share two facilities  $a, b$ .

<sup>2</sup>It is for instance the standard adjudication in bankruptcy situations, where the sum of creditors’ claims exceeds the liquidation value of the firm ([24]).

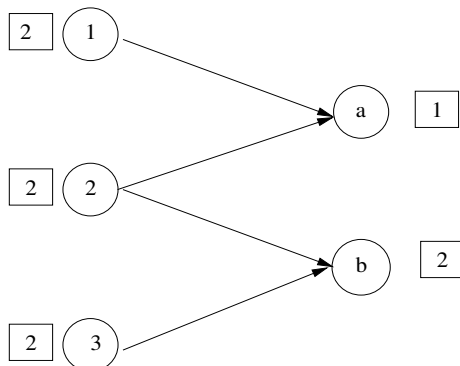


Figure 2: An example with 3 sources and 2 sinks

Agent 1 (resp. 3) is connected to facility  $a$  (resp.  $b$ ) only; agent 2 is connected to both facilities. Claims are identical  $x_1 = x_2 = x_3 = 2$ ; the capacities of facilities  $a$  and  $b$  are, respectively, 1 and 2 units. Notice again that the facilities are overdemanded. A max-flow assigns  $\varphi_{i\varepsilon}$  units of facility  $\varepsilon$  to agent  $i$  in such a way that  $\varphi_{1a} + \varphi_{2a} = 1$ ,  $\varphi_{2b} + \varphi_{3b} = 2$ . Agent 2's residual claim on  $a$  is  $2 - \varphi_{2b}$ , thus the division of that facility satisfies

$$\frac{\varphi_{1a}}{2} = \frac{\varphi_{2a}}{2 - \varphi_{2b}}$$

Similarly the division of facility  $b$  gives

$$\frac{\varphi_{2b}}{2 - \varphi_{2a}} = \frac{\varphi_{3b}}{2}$$

These four equations form a quadratic system with a unique solution

$$\varphi_{1a} = \frac{3}{\sqrt{7} + 2} = 0.646; \varphi_{2a} = \frac{2}{\sqrt{7} + 3} = 0.354 \quad (1)$$

$$\varphi_{2b} = \frac{6}{\sqrt{7} + 4} = 0.903; \varphi_{3b} = \frac{4}{\sqrt{7} + 1} = 1.097 \quad (2)$$

Agent 2 gets a smaller share of  $a$  than agent 1, and a smaller share of  $b$  than agent 3, yet his total share 1.257 is the largest, as announced.

Our main result says that a similar system delivers a unique *bipartite proportional* max-flow for any overdemanded problem on an arbitrary bipartite graph.

The proportional method is the most natural rationing method when there is a single sink node, but certainly not the only one. A substantial axiomatic literature (initiated in [34] and [2] and surveyed in [31] and [39]) discusses alternative methods, in particular two additional benchmark methods<sup>3</sup> with a

<sup>3</sup>The empirical social-psychology literature ([35], [17], [16]) confirms the central role of the three methods, proportional, uniform gains and uniform losses.

simple interpretation. To describe these methods, it is convenient to think of each source node as an agent, and its capacity as that agent’s claim. Similarly, we can think of the sink node as a resource, and its capacity as the amount available to be allocated to the agents. The *uniform gains* method equalizes individual shares as much as possible provided no one’s share exceeds his claim; the *uniform losses* method equalizes individual losses under the constraint that shares are non negative. In addition, a variety of methods provide flexible compromises between the three benchmarks: a good example is the family of *equal sacrifice* methods ([42], see section 7 below). We speak of a *standard* rationing method when there is a single sink node, and of a *bipartite* method in the case of multiple sink nodes.

The property known as *consistency* plays a central role in the axiomatic literature on standard rationing methods<sup>4</sup>. A method is consistent if, when we take away one agent from the set of participants, and subtract his share from the available resource, the division among the remaining set of claimants does not change. It is satisfied by the three benchmark methods, the family of equal sacrifice methods, and many more.

In bipartite rationing problems, we think of each sink node as a different “type” of resource, and the types of resource an agent can consume are perfect substitutes to satisfy his total demand. Now we can take away either a source node or a sink node, allowing us to generalize consistency to this more complex model. When we take away one agent, we subtract from each resource-type the share previously assigned to the departing agent; if we remove a resource-type, we subtract from the claim of each agent the share of the departing resource-type he was previously receiving; in each case we insist that the division in the reduced problem remain as before. The argument about residual claims in the examples of Figure 1 (resp. Fig. 2) is the instance of consistency applied to the removal of the resource-type *a* (resp. *a* then *b*).

We show how the other two benchmark methods—uniform gains and uniform losses—can be extended consistently to the bipartite context. Further, we show that many familiar consistent standard methods cannot be consistently extended to the bipartite context.

## 1.1 Interpreting Consistency

The interpretation of the connectivity constraints is critical to our model, and in particular to the Consistency axiom. Consider the substantial literature on maximum flow problems (see Ahuja et al. [1]) where the goal is to not only maximize the quantity distributed, but also to ensure that the distribution is equitable, which is the key motivation behind our model as well.

A typical example is the early contribution of Megiddo [28] who considers a network (not necessarily bipartite) with multiple sources and sinks and assumes that the manager wants “not only to maximize the total flow but also

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<sup>4</sup>Variants of this axiom have emerged in a variety of contexts, including TU games, matching, assignment, etc., as a compelling rationality property for fair division (see e.g., [27] and [40]). In the words of Balinski and Young: “every part of a fair division should be fair” [3].

to distribute it *fairly* among the sinks or the sources” (p.97). With the objective to “maximize the minimum amount delivered from individual sources” (ibid.), he proves the existence of a lexicographically optimal flow: among all flows maximizing the above minimum amount, it maximizes the second smallest amount delivered, etc<sup>5</sup>. Brown [10] discusses similarly the equitable distribution of coal during a prolonged coal strike in which only the 20-30% of “non-union” mines were active: the question is to distribute equitably the limited coal supply amongst the power companies with varying demands and connectivity constraints.<sup>6</sup>

Megiddo’s lex-optimal solution aims at equalizing flows going through the various sources and sinks, as much as permitted by the connectivity constraints: in the examples of Figure 1b and Fig. 2 where full equality of individual shares is possible, these constraints have no effect on final shares; e.g., in Figure 1b Bob is not penalized for having fewer connections than Ann.

We use the opposite normative postulate that agents should be held responsible for their connections.<sup>7</sup> In standard rationing models individual demands represent legal rights on the assets of a bankrupt firm ([34, 24]), or on the estate of a deceased person ([34, 2]), fiscal liability toward a levy ([43]), or any sort of objective “claim or liability” toward the resources. We generalize the standard model in that each individual claim applies to a subset of the resources, but we still require that the division of *each* resource-type be fair: the Consistency property achieves this by applying the same standard rationing method (for instance proportional) to each resource, and allowing each agent with a claim on this particular resource to apply her full *residual claim*, i.e., her initial claim minus her shares of other resources. This implies for instance that of two agents with identical claims, the one with richer connections carries a bigger total flow.

The *connection-neutral* view point a la Megiddo is entirely natural for applications such as famine relief, rationing of coal during a strike, rationing of blood of types O,A,B, or AB, among patients with these same types. An example where our *connection-responsible* viewpoint is compelling is the division of *earmarked funds* (as in [8]). Agents compete for the funds of several sponsors (federal agencies, private foundations, etc.); each agent submits a project with a total price tag, and each sponsor attaches some strings to the projects it will consider (e.g., must have an environmental dimension, must involve minorities, etc.); each project is submitted to all the sponsors of which it meets the constraints. Here the compatibility of my project with a given source of funds is anything but an act of God. Each sponsor seeks an equitable division of its own funds, which a consistent bipartite rationing method provides to all sponsors at the same time.

<sup>5</sup>Megiddo later gave an efficient algorithm to find a lex-optimal flow [29]; see also the work of Gallo, Grigoriadis, and Tarjan [21] for a more efficient implementation.

<sup>6</sup>It is not cost-effective (or feasible) to ship coal from certain mines to certain power plants.

<sup>7</sup>Modern theories of distributive justice (see [36, 19]) emphasize the distinction between personal characteristics for which individuals should be held responsible, and those for which they should not.

## 1.2 Related literature

We already mentioned the lex-optimal approach to fair maximum flows due to Megiddo, Brown, and a substantial extant literature.<sup>8</sup> In the same spirit of connection-neutrality, Minoux [30] considered a network with a single source and a single sink where each arc  $e$  cannot carry more than an  $\alpha(e)$  fraction of the total flow sent from the source ( $\alpha(e)$  is an exogenous parameter); his goal is to find a maximum flow that respects these constraints. This model was generalized by Zimmermann [44, 45], Hall and Vohra [22], and Betts and Brown [5] to allow for proportional *lower* and upper bounds on any arc, where as before the proportional bounds are increasing linear functions of the flow along one special arc (such as the total flow sent from the source). A typical application of this richer model is aid distribution during famine relief, where the proportional lower bounds ensure that no region receives too little of the total amount distributed.

Related optimization models such as the linear sharing problem [11] and the flow-circulation sharing problem [12] all address equitable distribution of resources in other settings, but again under connection-neutrality.

The work of Bochet et al. [8] is both more recent and closer in spirit to our work. In the same allocation problem with bipartite compatibility constraints between agents and resource types, they replace the objective *claims* of our model by privately held single-peaked *preferences* over one's total share, and assume the resources are non disposable. Thus, as in Sprumont's seminal model [38] with a single resource type, distributing all the resources typically requires to give some agents more than their peak allocation, and some less. The connection-neutral extension of Sprumont's uniform gains method selects the Lorenz dominant feasible profile of total shares (it coincides with Megiddo's lex-optimal solution). The corresponding direct revelation mechanism is strategyproof<sup>9</sup> (even group-strategyproof: see [13]), a characteristic property under connection-neutrality.

Bochet et al. [9] is a variant of [8] with strategic agents on both sources and sinks, the source-agents demanding some resource up to some privately held peak level, while the sink agents want to supply resource up to their own peak level. Efficient trade splits the market in a segment where suppliers get their peak allocation while the relevant demanders are rationed, and another segment where the roles are reversed. These authors maintain connection-neutrality and focus as before on the Lorenz dominant efficient trade.

Random assignment under dichotomous preferences, studied by Bogomolnaia and Moulin [7] and Roth et al. [37], is the special case of [9] where all claims are for one unit and there is one unit of each resource-type. In that model as here, the assumption of unit claims and unit types does not significantly simplify the computations.

Finally Ilkilic and Kayi [23] discuss a bipartite rationing model with objective claims and resources like we do here, but under connection neutrality.

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<sup>8</sup>See the survey by Luss [26].

<sup>9</sup>Each agent reports his preferences and the truthful report is a dominant strategy.

They construct in that spirit reasonable extensions of general standard rationing methods.

Our work is also loosely related to some recent papers on bargaining and networks. Inspired by the network exchange theory from sociology, Kleinberg & Tardos [25] and Chakraborty et al. [14, 15] develop models of bargaining on networks where each node-agent engages in bilateral negotiations with other node-agents to which he is connected on a fixed graph. The division problem is quite different in [25] than in ours because each agent can strike only one deal. But in [14], [15], each pair of connected agents strike a bargain to share their pair-specific surplus. This is like in the special case of our model where each resource-type is connected to exactly two agents, and represents the amount of surplus over which these two agents bargain. Then agent  $i$ 's disagreement point in his negotiation with  $j$  is determined by the sum of his shares in all other bilateral negotiations. Given an exogenous bargaining rule for two-person problems, an equilibrium profile of bilateral surplus divisions is defined by a consistency property formally similar to ours. However the qualitative effect is exactly opposite: in [14, 15], the bigger my disagreement outcome, the larger my share of the surplus, whereas in our model a bigger share of resource-types other than  $a$  decreases my claim on, and my share of  $a$ . The intersection of the two models is the uninteresting case with linear utility and very large equal claims, so that each pairwise surplus is divided equally, irrespective of the graph.

### 1.3 Overview of our results

We define bipartite rationing problems and methods in section 2, and our most basic axioms in section 3: we restrict attention to rationing methods that are symmetric (the labeling of agents and resources does not matter), continuous (the maxflow as a function of demands and resources endowments), and treat all resource-types as a single type when the bipartite graph is complete (everyone can consume every type). We define two versions of Consistency in section 4, with respect to nodes, or to edges: when we remove a certain edge, we subtract its flow from the capacity of both end nodes, and require that the solution choose the same flow in the reduced problem. We are looking for *standard* rationing methods that can be extended to a *consistent bipartite* method.

Our main result (Theorem 1 in section 5) is that the standard proportional method is uniquely extendable. Its extension can be described in two equivalent ways. For problems such that every subproblem is strictly overdemanding, the method assigns a unique set of convex weights  $w_i$  to the agents and divides each resource-type in proportion to the  $w_i$ s of the agents who can consume this resource; moreover individual *losses* (claim minus total share of an agent over all resources he can consume) are proportional to the  $w_i$ -s as well. The weights are *not* proportional to the individual claims. An alternative definition is that the proportional method minimizes the sum of two entropies, that of a max-flow plus that of the corresponding profile of losses.

We show in section 6 that the uniform gains and uniform losses methods are also extendable, however unlike the proportional, each method admits infinitely

many consistent extensions to the bipartite context (Propositions 1,2).

In section 7 we state a critical necessary condition for a standard method to be consistently extendable to the bipartite context (Lemma 2). If we distribute  $t\%$  of the final shares and reduce claims and resources accordingly, then in the smaller problem everyone gets the remaining  $(100 - t)\%$  of his original share<sup>10</sup>. We use this technical property to deduce that many familiar rationing methods are *not* extendable as desired. Examples include the Talmudic ([2]) and most equal sacrifice ([43]) methods. The companion paper [33] establishes that this necessary condition is essentially sufficient, and discusses the new class of standard rationing methods it identifies.

In section 8 we list some open questions that merit further study. Finally the Appendix states a decomposition result (Lemma 3) that is useful throughout the paper.

## 2 Model and Notation

We have a set  $\mathcal{N}$  of potential agents and a set  $\mathcal{Q}$  of potential resource-types (or simply types). An *instance* of the rationing problem is obtained by first picking a set  $N$  of  $n$  agents, a set  $Q$  of  $q$  types, and a bipartite graph  $G \subseteq N \times Q$ ; an edge  $(i, a) \in G$  indicates that agent  $i$  can consume the type  $a$ . We do not assume that  $G$  is connected. We define  $f(i)$  to be the set of types that  $i$  is connected to, and  $g(a)$  to be the set of agents that connect to type  $a$ . That is,  $f(i) = \{a \in Q \mid (i, a) \in G\}$  and  $g(a) = \{i \in N \mid (i, a) \in G\}$ . We assume that  $f(i)$  and  $g(a)$  are non-empty for each  $i$  and  $a$ .

Next, each agent  $i$  has a claim  $x_i$  and each type  $a$  has a capacity (amount it can supply)  $r_a$ ; these are arbitrary non-negative numbers. We let  $x$  be the vector of claims and  $r$  be the vector of resource capacities. For a subset  $B$  and a vector  $y$ , we use the notation  $y_B := \sum_{i \in B} y_i$ . Also, for vectors  $y$  and  $z$ ,  $y \ll z$  stands for  $y_i < z_i$  for all  $i$ .

A bipartite *flow* problem is specified by  $P = (N, Q, G, x, r)$  or simply  $P = (G, x, r)$  if the sets  $N$  and  $Q$  are clear from context.

Given a flow problem  $P$ , a flow  $\varphi$  specifies a non-negative real number  $\varphi_{ia}$  for each edge  $(i, a)$  in  $G$  such that

$$\varphi_{g(a)a} \leq r_a \text{ for all } a \in Q; \text{ and } \varphi_{if(i)} \leq x_i \text{ for all } i \in N,$$

where  $\sum_{i \in g(a)} \varphi_{ia} \stackrel{\text{def}}{=} \varphi_{g(a)a}$ , and  $\sum_{a \in f(i)} \varphi_{ia} \stackrel{\text{def}}{=} \varphi_{if(i)}$ . The flow  $\varphi$  is a *max-flow* if it maximizes  $\sum_i \varphi_{if(i)}$  (equivalently  $\sum_a \varphi_{g(a)a}$ ). Define  $F(P)$ , or  $F(G, x, r)$ , to be the set of max-flows for problem  $P = (G, x, r)$ ; any  $\varphi \in F(P)$  is called a solution to the problem  $P$ . Agent  $i$ 's total transfer  $y_i = \varphi_{if(i)}$  is called his allocation, or share. Although agents care only about their allocation, not its flow decomposition, we must nevertheless work with flows, on which our key axioms bear.

<sup>10</sup>This property is in the spirit of, though not logically related to, Consistency and the Lower composition axiom (see Moulin [31] and Thomson [39]).



We now make a simple observation that lets us assume additional structure on any flow problem without loss of generality. A familiar consequence of the max-flow min-cut theorem ([1]) is that we can decompose any max-flow problem in (at most) two simpler subproblems that can be treated separately. In one subproblem the sink nodes are overdemanded, in the sense that in every solution  $\varphi$ , these resource-types are fully allocated to the underdemanded agents, each of whom receives at most his claim; so these agents are rationed. The situation is reversed in the other subproblem, where, in every solution  $\varphi$ , the overdemanded agents receive exactly their claim from the underdemanded sink nodes. Because there is no edge between two underdemanded nodes, this decomposition cuts our fair division problem in half: we need only to propose a rule for problems where the sinks are overdemanded and the sources rationed, then exchange the role of sources and sinks to apply the same rule to problems with overdemanded sources and rationed sinks.

In the rest of the paper, we shall be concerned only with problems in which the resources are overdemanded. It is well known (see [1] or [9]) that the system of inequalities (3), shown below, characterizes the existence of a flow  $\varphi$  exhausting all resources and transferring at most his claim to each agent  $i$ .

**Definition 1** *A bipartite rationing problem is a flow problem  $P = (N, Q, G, x, r)$  such that the resources are overdemanded, namely:*

$$\text{for all } B \subseteq Q: r_B \leq x_{g(B)}. \quad (3)$$

Let  $\mathcal{P}$  denote the set of bipartite rationing problems  $P = (G, x, r)$ .

Three subsets of  $\mathcal{P}$  play an important role below. A problem  $P \in \mathcal{P}$  is *strictly overdemanded* if

$$\text{for all } B \subseteq Q: r_B < x_{g(B)}.$$

Let  $\mathcal{P}^{str}$  be the set of strictly overdemanded problems. A problem  $P \in \mathcal{P}$  is *irreducible* if every subproblem is strictly overdemanded:

$$r_Q \leq x_N; \text{ for all } B \subsetneq Q: r_B < x_{g(B)}.$$

Let  $\mathcal{P}^{ir}$  be the set of irreducible problems.

Finally, a  $P \in \mathcal{P}$  is *balanced* if  $r_Q = x_{g(Q)}$ .

Note that a problem  $P \in \mathcal{P} \setminus \mathcal{P}^{ir}$  must contain a balanced subproblem, and so can be further decomposed: focusing on the balanced subproblem, observe that the resources involved are enough to satisfy every agent involved in the balanced subproblem, so such agents receive nothing from the resources outside of the subproblem. By iteratively eliminating such balanced subproblems, we end up with at most one irreducible problem. This is the key to the canonical decomposition of an arbitrary problem in  $\mathcal{P}$  into a union of irreducible problems, all but at most one of them balanced: see Lemma 3 in the Appendix.

Note further that an irreducible and balanced problem must have a connected graph, however a strictly overdemanded problem need not be connected.

**Definition 2** A bipartite rationing method (or simply method)  $H$  associates to each problem  $P \in \mathcal{P}$ , where  $N \subset \mathcal{N}, Q \subset \mathcal{Q}$ , a max-flow  $\varphi = H(P) \in \mathcal{F}(P)$ .

Note that any agent with zero claim, and any type with zero resource gets no flow in any method.

**Definition 3** A rationing problem is standard if it involves a single resource type to which all agents are connected. It is a triple  $P^0 = (N, x, t)$ , where  $x \in \mathbb{R}_+^N$  is the profile of claims,  $t$  units of the resource are available, and  $t \leq x_N$ . We write  $\mathcal{P}^0$  for the set of standard problems.

A standard rationing method  $h$  is a method applying only to standard problems. Thus  $h(N, x, t) \in \mathbb{R}_+^N$  is a division of  $t$  among the agents in  $N$  such that  $h_i(N, x, t) \leq x_i$  for all  $i \in N$ .

We recall the definition of the three benchmark standard rationing methods, proportional  $h^{pro}$ , uniform gains  $h^{ug}$ , uniform losses  $h^{ul}$ :

$$h^{pro}(x, t) = \frac{x_i}{x_N} \cdot t;$$

$$h_i^{ug}(x, t) = \min\{x_i, \lambda\} \text{ where } \lambda \text{ solves } \sum_{i \in N} \min\{x_i, \lambda\} = t;$$

$$h_i^{ul}(x, t) = \max\{x_i - \mu, 0\} \text{ where } \mu \text{ solves } \sum_{i \in N} \max\{x_i - \mu, 0\} = t.$$

For each resource  $a \in Q$ , a method  $H$  defines a standard rationing method  ${}^a h$  by the way it deals with this single resource and the complete graph  $G = N \times \{a\}$ :

$${}^a h(N, x, r_a) = H(N \times \{a\}, x, r_a)$$

### 3 Basic axioms

As discussed in the introduction, our goal is to understand which standard methods can be extended to bipartite methods, while respecting a consistency property. As in most of the literature on standard methods (see e.g., [31], [39]), we restrict attention to symmetric and continuous rationing methods.

**Symmetry (SYM).** A method  $H$  is *symmetric* if the labels of the agents and types do not matter. Formally, given a permutation  $\pi$  of the agents and a permutation  $\sigma$  of the types, define  $G^{\pi, \sigma}$  to be the graph such that  $(\pi(i), \sigma(a)) \in G^{\pi, \sigma}$  if and only if  $(i, a) \in G$ . The claims  $x^\pi$  of the agents and resources  $r^\pi$  of the types are similarly defined. Suppose  $H(G, x, r) = \varphi$  and  $H(G^{\pi, \sigma}, x^\pi, r^\pi) = \varphi'$ . Then the method  $H$  is symmetric if and only if  $\varphi_{ia} = \varphi'_{\pi(i)\sigma(a)}$  for all  $(i, a) \in G$ .

The standard method associated with a symmetric  $H$  is symmetric as well, thus independent of the choice of the type  $a$  and the agents  $N$ . In keeping with the rest of our notation, we write it simply as  $h(x, t)$ , where  $x \rightarrow h(x, t)$  is symmetric from  $\mathbb{R}_+^n$  into itself.

**Continuity (CONT).** A method  $H$  is *continuous* if for all  $N, Q$ , and  $G$ , the mapping  $(x, r) \rightarrow H(G, x, r)$  is continuous in the relevant subset of  $\mathbb{R}_+^N \times \mathbb{R}_+^Q$ .

We also insist that our methods do not distinguish a problem without any compatibility constraints (i.e., the graph  $G$  is complete) from the corresponding standard problem where all types are merged into one.

**Reduction of Complete Graphs (RCG).** Fix a problem  $P = (N \times Q, x, r) \in \mathcal{P}$  where the graph  $G$  is complete. The symmetric method  $H$ , with associated standard method  $h$ , satisfies RFG if for all  $N \subset \mathcal{N}, Q \subset \mathcal{Q}$ , and all  $(N \times Q, x, r) \in \mathcal{P}$ , we have

$$\varphi_{iQ} = h(x, r_Q) \quad (4)$$

i.e., the shares  $y(P)$  obtain by merging all resources into a single type.

**Definition 4** We write  $\mathcal{H}^0$  for the set of symmetric and continuous standard rationing methods, and  $\mathcal{H}$  for the set of symmetric, continuous bipartite methods satisfying Reduction of Complete Graphs. We use the notation  $\mathcal{H}(A, B, \dots), \mathcal{H}^0(A, B, \dots)$  for the subset of methods in  $\mathcal{H}$  or  $\mathcal{H}^0$  satisfying additional properties  $A, B, \dots$ .

## 4 Consistency

We give two versions of the crucial consistency property, both generalizing consistency for standard methods.

We use the following notation. For a given graph  $G \subseteq N \times Q$ , and subsets  $N' \subseteq N, Q' \subseteq Q$ , the *restricted graph* of  $G$  is  $G(N', Q') := G \cap \{N' \times Q'\}$ , again not necessarily connected, and the *restricted problem* obtains by also restricting  $x$  to  $N'$  and  $r$  to  $Q'$ .

**Node Consistency (Node-CSY).** Fix an agent  $i \in N$  and a problem  $P \in \mathcal{P}$ , and define the reduced claims and resources under method  $H \in \mathcal{H}$  after this agent (and all the edges involving this agent) is removed:

$$x_j^H(-i) = x_j, \text{ for all } j \neq i$$

and for  $\varphi = H(P)$ :

$$r_a^H(-i) = r_a - \varphi_{ia} \text{ for all } a \in f(i); \quad r_b^H(-i) = r_b, \text{ for } b \notin f(i).$$

The reduced problem is  $(G(N^*, Q^*), x^H(-i), r^H(-i))$ , where  $N^* = N \setminus \{i\}$ , and  $Q^* = f(N^*)$ . Similarly, fix a type  $a \in Q$  and define the reduced claims and resources under method  $H$  after this type (and all the edges involving this type) is removed:

$$x_j^H(-a) = x_j - \varphi_{ja} \text{ for all } j \in g(a); \quad x_j^H(-a) = x_j, \text{ for } j \notin g(a).$$

and

$$r_b^H(-a) = r_b, \text{ for all } b \neq a$$

The reduced problem is  $(G(N^{**}, Q^{**}), x^H(-a), r^H(-a))$ , where  $Q^{**} = Q \setminus \{a\}$ ,  $N^{**} = g(Q^{**})$ .

Suppose  $H(G(N, Q), x, r) = \varphi$ ,  $H(G(N^*, Q^*), x^H(-i), r^H(-i)) = \varphi'$ , and  $H(G(N^{**}, Q^{**}), x^H(-a), r^H(-a)) = \varphi''$ . The method  $H \in \mathcal{H}$  is *node-consistent* if for all  $N \subset \mathcal{N}, Q \subset \mathcal{Q}$ , all  $(G, x, r) \in \mathcal{P}$ , all  $i \in N, a \in Q$ :  $\varphi_{jb} = \varphi'_{jb}$  for all  $jb \in G(N^*, Q^*)$  and  $\varphi_{jb} = \varphi''_{jb}$  for all  $jb \in G(N^{**}, Q^{**})$ .

**Edge Consistency (Edge-CSY).** *Edge-consistency* is stronger than node-consistency. Fix an edge  $ia \in G$  and define the *reduced claims and resources* under method  $H$  after this edge is removed:

$$x_i^H(-ia) = x_i - \varphi_{ia}; \quad x_j^H(-ia) = x_j \text{ for } j \neq i$$

$$r_a^H(-ia) = r_a - \varphi_{ia}; \quad r_b^H(-ia) = r_b \text{ for } b \neq a$$

The corresponding reduced problem is  $(G \setminus \{ia\}, x^H(-ia), r^H(-ia))$ , where the set of agents is  $N^* = N$  unless  $f(i) = \{a\}$  in which case  $N^* = N \setminus \{i\}$ ; similarly the set of types is  $Q^* = Q$  unless  $g(a) = \{i\}$  in which case  $Q^* = Q \setminus \{a\}$ .

Suppose  $H(G, x, r) = \varphi$  and  $H(G \setminus \{ia\}, x^H(-ia), r^H(-ia)) = \varphi'$ . The method  $H \in \mathcal{H}$  is *edge-consistent* if for all  $N \subset \mathcal{N}, Q \subset \mathcal{Q}$ , all  $(G, x, r) \in \mathcal{P}$ , and  $ia \in G$ :  $\varphi_{jb} = \varphi'_{jb}$  for all  $jb \in G \setminus \{ia\}$ .

Clearly, for either one of the three reductions just discussed, the reduced problem is overdemandd if the initial problem is, but not necessarily strictly overdemandd or irreducible if the initial problem is. Note also that  $G \setminus \{ia\}$  may not be connected even if  $G$  is connected.

## 5 The bipartite proportional method

Given the prominent role of the standard proportional method in  $\mathcal{H}^0$ , the first question is to look for a bipartite extension. It turns out that there is a unique such extension  $H^{pro}$  satisfying Node-CSY.

Theorem 1 gives two equivalent definitions of this method, one for any overdemandd problem as the solution of a maximization problem, the other for irreducible problems only. The latter definition is then extended to any overdemandd problem by means of its canonical decomposition in irreducible subproblems (Definition 5 in the Appendix). The latter definition gives much more insight into the structure of our method.

We use two new pieces of notation. The unit simplex of  $\mathbb{R}^N$  is written below as  $\mathcal{S}(N)$ , and its interior as  $\mathring{\mathcal{S}}(N) = \{w | w_N = 1 \text{ and } w_i > 0 \text{ for all } i\}$ . For any  $z \geq 0$ , we define the function  $En(z) = z \ln(z)$ , with the convention that  $En(0) = 0$ . Note that the sum  $\sum_k En(z_k)$  is the familiar *entropy* of a vector  $z$ . Note also that  $En(z)$  is strictly convex.

For any problem  $P = (G, x, r) \in \mathcal{P}$ , define  $\widehat{\varphi}(P)$  as

$$\widehat{\varphi}(P) = \arg \min_{\varphi \in \mathcal{F}(G, x, r)} \sum_{ia \in G} En(\varphi_{ia}) + \sum_{i \in N} En(x_i - \phi_{if(i)}) \quad (5)$$

Problem (5) has a unique solution  $\widehat{\varphi}$  for any  $P \in \mathcal{P}$  because the objective function is strictly convex and finite. This defines the *proportional method*,  $H^{pro}$ , which associates with each problem  $P$  the solution  $\widehat{\varphi}(P)$ , that we call the *proportional flow* for  $P$ .

The following result establishes additional properties of the proportional method and provides an alternative definition of the proportional flow.

**Theorem 1**

- i) *The proportional method  $H^{pro}$  is in  $\mathcal{H}$  and is edge-consistent:  $H^{pro} \in \mathcal{H}(\text{Edge} - \text{CSY})$ .*
- ii) *For any irreducible problem  $P = (G, x, r) \in P^{ir}$ , the following system with unknown  $w \in S(N)$*

$$x_i = w_i \cdot (x_N - r_Q) + \sum_{a \in f(i)} \frac{w_i}{w_{g(a)}} r_a \text{ for all } i \in N \quad (6)$$

has a unique solution  $\widehat{w}$  in  $\overset{\circ}{S}(N)$ , and the proportional flow is

$$\widehat{\varphi}_{ia} = \frac{\widehat{w}_i}{\widehat{w}_{g(a)}} r_a \quad (7)$$

- iii) *The method  $H^{pro}$  is the only continuous and node-consistent method that is proportional for standard problems.*

For instance the example of Figure 2 is irreducible, and the system (7) writes

$$\varphi_{ia} = \frac{w_i}{w_1 + w_2} \cdot 1 \text{ for } i = 1, 2; \varphi_{ib} = \frac{w_i}{w_2 + w_3} \cdot 2 \text{ for } i = 2, 3$$

Moreover (6) gives  $x_i - y_i = w_i \cdot (x_N - r_Q)$ , i.e.,

$$\frac{2 - \varphi_{1a}}{w_1} = \frac{2 - (\varphi_{2a} + \varphi_{2b})}{w_2} = \frac{2 - \varphi_{3b}}{w_3} = 3$$

The unique solution

$$w_1 = \frac{1}{3}(4 - \sqrt{7}); w_2 = \frac{1}{9}(5\sqrt{7} - 11); w_3 = \frac{2}{9}(4 - \sqrt{7})$$

confirms the flow (1), (2) found in section 1.

Equation (7) explains our proportional terminology for the method  $H^{pro}$ . Indeed the flow  $\widehat{\varphi}$  distributes each resource  $a$  proportionally between the agents connected to  $a$ , however the proportionality is not with respect to the original claims, but with respect to the “adjusted claims”  $w$ . The adjustments account for the asymmetry in the connections of the various agents. The relationship between the adjusted claims and the original claims is given in Equation (6).

**Proof of Theorem 1**

We first argue that  $H^{pro}$  is symmetric and continuous. It is clear that any relabeling of the agents and resources does not change the optimization problem

characterizing the proportional solution, so symmetry follows immediately. The fact that  $H^{pro}$  is continuous follows from Berge's Maximum Theorem ([6]): The objective function is continuous, and the correspondence  $(x, r) \rightarrow \mathcal{F}(G, x, r)$  is compact-valued, and continuous as well (upper and lower hemicontinuous); therefore the argmin correspondence is continuous as well.

We observe immediately after this proof that  $H^{pro}$  satisfies a property stronger than RCG, dubbed Merging of Identical Resource-types, so a direct proof of RCG is not needed at this point.

*Step 1: Statement i)* For Edge-CSY, we fix  $P = (G, x, r)$  and an edge  $ia \in G$ . For any  $\varphi' \in \mathcal{F}(G \setminus \{ia\}, x^H(-ia), r^H(-ia))$ , adding  $ia$  to  $G$  and  $\widehat{\varphi}_{ia}$  to  $\varphi'$  yields a flow  $(\varphi', \widehat{\varphi}_{ia})$  in  $\mathcal{F}(G, x, r)$ . The objective function at  $(\varphi', \widehat{\varphi}_{ia})$  is the same as at  $\varphi'$  plus the single term  $En(\widehat{\varphi}_{ia})$ , because  $x_i - y_i = x_i^H(-ia) - y_i^H(-ia)$ . Thus if the restriction of  $\widehat{\varphi}$  to  $P^H(-ia)$  is not optimal in that problem, we can construct a flow  $(\varphi', \widehat{\varphi}_{ia})$  beating  $\widehat{\varphi}$  in  $P$ .

*Step 2: Statement ii)* We fix an irreducible problem  $P = (G, x, r)$ . It will be convenient to replace problem (5) by the equivalent problem

$$\min_{\varphi \in \mathcal{F}(G, x, r)} \sum_{ia \in G} Ln(\varphi_{ia}) + \sum_{i \in N} Ln(x_i - \varphi_{if(i)}) \quad (8)$$

where  $Ln(z) = z(\ln(z) - 1)$  is still strictly convex and has derivative  $\ln(z)$ . The equivalence follows from the fact that we are subtracting two constant terms to the objective function:  $\sum_{ia \in G} \varphi_{ia} = r_Q$  and  $\sum_{i \in N} (x_i - \varphi_{if(i)}) = x_N - r_Q$ .

*Step 2.1* We assume in this sub-step  $x_N = r_Q$ :  $P$  is balanced. By irreducibility, for every  $(i, a) \in G$ , there is a solution  $\varphi \in \mathcal{F}(G, x, r)$  with  $\varphi_{ia} > 0$ . Also, because the problem is balanced,  $\varphi_{if(i)} = x_i$  for every  $\varphi \in \mathcal{F}(G, x, r)$ . Thus Problem (8) becomes

$$\min_{\varphi \in \mathcal{F}(G, x, r)} \sum_{ia \in G} Ln(\varphi_{ia}),$$

whose Lagrangean is given by

$$L(\varphi, \lambda, \mu) = \sum_{(i,a) \in G} \varphi_{ia} [\ln(\varphi_{ia}) - 1] + \sum_{i \in N} \lambda_i (x_i - \sum_{a \in Q} \varphi_{ia}) + \sum_{a \in Q} \mu_a (r_a - \sum_{i \in N} \varphi_{ia}),$$

where  $\lambda = (\lambda_i)_{i \in N} \in \mathbb{R}_+^N$  and  $\mu = (\mu_a)_{a \in Q} \in \mathbb{R}^Q$ . Define

$$q(\lambda, \mu) = \min_{\varphi \geq 0} L(\varphi, \lambda, \mu). \quad (9)$$

It is easy to check that for any fixed  $\lambda$  and  $\mu$ , the minimum is attained in (9) uniquely by the solution  $\varphi_{ia}^* = e^{\lambda_i + \mu_a}$ , using which we get

$$q(\lambda, \mu) = L(\varphi^*, \lambda, \mu) = - \sum_{(i,a) \in G} \varphi_{ia}^* + \sum_{i \in N} \lambda_i x_i + \sum_{a \in Q} \mu_a r_a.$$

The associated dual problem is thus given by

$$\max_{\lambda, \mu} \left\{ - \sum_{(i,a) \in G} e^{\lambda_i + \mu_a} + \sum_{i \in N} \lambda_i x_i + \sum_{a \in Q} \mu_a r_a \right\}. \quad (10)$$

It is clear that (10) has a *unique* optimal solution that is given by the solution to the following system of equations:

$$- \sum_{a \in f(i)} e^{\lambda_i + \mu_a} + x_i = 0, \quad \forall i \in N,$$

and

$$- \sum_{i \in g(a)} e^{\lambda_i + \mu_a} + r_a = 0, \quad \forall a \in Q.$$

Letting  $\lambda^*$  and  $\mu^*$  be the optimal solutions, we have

$$e^{\lambda_i^*} = \frac{x_i}{\sum_{a \in f(i)} e^{\mu_a^*}}; \quad e^{\mu_a^*} = \frac{r_a}{\sum_{i \in g(a)} e^{\lambda_i^*}}.$$

Finally,

$$\varphi_{ia}^* = e^{\lambda_i^*} e^{\mu_a^*}.$$

In particular, taking  $w_i = \frac{e^{\lambda_i^*}}{\sum_N e^{\lambda_j^*}}$  verifies (6) and (7).

*Step 2.2* We assume now that  $P = (G, x, r)$  is not only irreducible, but also strictly overdemand, i.e.  $x_N > r_Q$ . We proceed as before by writing the Lagrangean of Problem (8), which is now

$$\begin{aligned} L(\varphi, \lambda, \mu) &= \sum_{(i,a) \in G} Ln(\varphi_{ia}) + \sum_{i \in N} Ln\left(x_i - \sum_{a \in f(i)} \varphi_{ia}\right) \\ &+ \sum_{i \in N} \lambda_i (x_i - \sum_{a \in Q} \varphi_{ia}) + \sum_{a \in Q} \mu_a (r_a - \sum_{i \in N} \varphi_{ia}). \end{aligned}$$

As before, for any fixed  $\lambda$  and  $\mu$ , the minimum in the problem

$$q(\lambda, \mu) = \min_{\varphi \geq 0} L(\varphi, \lambda, \mu)$$

is attained uniquely by the solution of

$$\frac{\varphi_{ia}^*}{x_i - \sum_{a \in f(i)} \varphi_{ia}^*} = e^{\lambda_i + \mu_a}.$$

An implication of this is that in the minimizer of  $q(\lambda, \mu)$ , each agent's allocation  $y_i$  is such that  $y_i < x_i$ . This implies that the optimal choice of  $\lambda$  in the associated dual problem  $\max_{\lambda \geq 0, \mu} q(\lambda, \mu)$  is  $\lambda^* = 0$ . Also, it is straightforward to check that the dual is a maximization problem with a strictly concave objective function, and so has a unique optimal solution  $\mu^*$ . Using this, the optimal  $\varphi_{ia}^*$  satisfy  $(x_i - \sum_{b: b \in f(i)} \varphi_{ib}^*) e^{\mu_a^*} = \varphi_{ia}^*$ . Letting  $y_i^* = \sum_{b: b \in f(i)} \varphi_{ib}^*$ , we see, in particular, that

$$\frac{\varphi_{ia}^*}{x_i - y_i^*} = \frac{\varphi_{ja}^*}{x_j - y_j^*} = \frac{r_a}{x_{g(a)} - y_{g(a)}^*}, \quad \text{for all } a \text{ and } i, j \in g(a) \quad (11)$$

Setting  $\widehat{w}_i = \frac{x_i - y_i^*}{x_N - r_Q}$ , so that  $\widehat{w} \in \overset{\circ}{\mathcal{S}}(N)$ , we see that  $\widehat{w}$  is a solution of system (6). Moreover (11) implies (7) as well.

*Step 3: Statement iii)*

Let  $H$  be a continuous and node consistent method, proportional for standard problems. Pick first a strictly overdemanded  $P = (G, x, r) \in \mathcal{P}^{str}$ . Fix a type  $a$  and reduce  $P$  by dropping successively all nodes but  $a$ . Then Node-CSY and the fact that  $H$  is proportional for one-type problems imply:

$$\text{for all } i \in g(a) : \varphi_{ia} = h^{pro}(x - y + \varphi_{\cdot a}, r_a) = \frac{x_i - y_i + \varphi_{ia}}{x_{g(a)} - y_{g(a)} + r_a} r_a; \text{ or } \varphi_{ia} = 0 \quad (12)$$

If  $y_i = x_i$  this implies either  $\varphi_{ia} = 0$  or  $\{\varphi_{ia} = r_a \text{ and } \{y_j = x_j \text{ and } \varphi_{ja} = 0 \text{ for all } j \in g(a)\}\}$ . Restricting attention to a connected component of  $G$ , this implies that every resource goes to a single agent and they all have  $y_j = x_j$ , contradiction.

So  $y_i < x_i$  for all  $i$ . Then (12) implies  $\varphi_{ia} > 0$  for all  $ia \in G$ . It also reduces to

$$\varphi_{ia} = \frac{x_i - y_i}{x_{g(a)} - y_{g(a)}} r_a \Rightarrow \frac{\varphi_{ia}}{x_i - y_i} = \frac{\varphi_{ja}}{x_j - y_j} = \frac{r_a}{x_{g(a)} - y_{g(a)}} \text{ for all } i, j \in g(a)$$

These are precisely the KKT optimality conditions, so  $\varphi = \varphi^*$ .

Pick next  $P = (G, x, r)$  irreducible and balanced. Both  $H$  and the proportional bipartite method  $H^{pro}$  are continuous, and  $P$  can be expressed as the limit of strictly overdemanded problems<sup>11</sup>. Thus  $H = H^{pro}$  on  $\mathcal{P}^{ir}$ .

Finally, both methods  $H$  and  $H^{pro}$  are node-consistent on  $\mathcal{P}$ , so as explained after Definition 6 in the Appendix, they are the canonical extension of their projection on  $\mathcal{P}^{ir}$ , where they coincide.

We note that the proof of statement *iii)* only requires to assume consistency with respect to the elimination of resource-types. ■

The proportional method, as well as all methods discussed in the next two Propositions, satisfies a (much) stronger property than Reduction of Complete Graphs: if two resource-types are compatible with exactly the same set of agents, they need not be treated as separate types in the sense that merging them into a single type while adding their resources is of no consequence to any agent. Thus the artificial creation of new resource-types does not matter.

**Merging Identically Connected Resource-types (MIR).** Fix a problem  $P \in \mathcal{P}$  and suppose that in the graph  $G \subseteq N \times Q$ , two types  $a_1, a_2$  are such that  $g(a_1) = g(a_2)$ . Let  $G^* \subseteq N \times (Q \setminus \{a_1, a_2\} \cup \{a^*\})$  be the graph obtained by *merging* those two types into a new node labeled  $a^*$  with the same connections. The corresponding *merged* problem  $(G^*, x, r^*)$  has  $r_{a^*}^* = r_{a_1} + r_{a_2}$ ,  $r_a^* = r_a$  for all  $a \in Q \setminus \{a_1, a_2\}$ .

<sup>11</sup>Consider the sequence of problems  $P^\epsilon = (G, x^\epsilon, r)$  with  $x_i^\epsilon = x_i + \epsilon$  for all  $i$ , and let  $\epsilon \rightarrow 0$ . For every  $\epsilon > 0$ ,  $P^\epsilon$  is strictly overdemanded.



Suppose  $H(G, x, r) = \varphi$  and  $H(G^*, x, r^*) = \varphi^*$ . The method  $H \in \mathcal{H}$  allows the *merging of identically connected types* if for all  $N \subset \mathcal{N}, Q \subset \mathcal{Q}$ , all  $(G, x, r) \in \mathcal{P}$ , and  $a_1, a_2$  s.t.  $g(a_1) = g(a_2)$ :  $\varphi_{ia^*}^* = \varphi_{ia_1} + \varphi_{ia_2}$  for all  $i \in g(a^*)$ ,  $\varphi_{ja^*}^* = \varphi_{ja}$  for all  $a \in Q \setminus \{a_1, a_2\}$ ,  $ja \in G$ . In particular individual shares  $y_i$  are unchanged.

To check that  $H^{pro}$  satisfies MIR, we fix an irreducible problem  $(G, x, r)$  with weights  $w_i = \frac{x_i}{x_N}$  solving (6), and assume  $g(a_1) = g(a_2)$ . After merging  $a_1$  and  $a_2$  into  $a$ , the weights  $\tilde{w}_a = w_{a_1} + w_{a_2}$ ,  $\tilde{w}_b = w_b$  for  $b \neq a_1, a_2$ , satisfy the corresponding system (6) in the merged problem, so statement  $i$ ) implies that MIR holds in  $\mathcal{P}^{ir}$ . For a general  $(G, x, r) \in \mathcal{P}$  we use its canonical decomposition in irreducible problems (Lemma 3 in the Appendix): clearly two nodes such that  $g(a_1) = g(a_2)$  must be in the same component  $Q^k$  of the decomposition, where MIR applies, and the merging of these two nodes reduce  $Q^k$  by one type and preserves the rest of the decomposition.

We can formulate an axiom parallel to MIR for the merging of agents. When two agents  $i, j$  have identical connections,  $f(i) = f(j)$ , we can merge them into a single agent, and endow this superagent with the sum of their claims. The corresponding “merging of identically connected agents” (MIA) property says that the flow in all edges not involving  $i$  or  $j$  must be unchanged, while the flow in the merged edges is the sum of the two earlier flows.

This property is known to force the proportional method for standard problems ([4], [32]), and in the bipartite context it takes us uniquely to its canonical extension: system (6) implies at once that  $H^{pro}$  satisfies MIA. This yields an alternative characterization of the bipartite proportional method, by continuity, node (resource-types) consistency, and merging of identically connected agents.

The critical difference between MIR and MIA is that the latter applies exclusively to the bipartite proportional method, whereas the former holds true for many more consistent bipartite methods, such as all the extensions of uniform gains and uniform losses in the next section, and all loss calibrated methods discussed in [33].

We conclude this section with one more agreeable feature of  $H^{pro}$ : if agents  $i, j$  have identical claims but  $i$  is better connected, she gets a weakly bigger share than  $j$ . This is illustrated by the examples in Figures 1,2 (section 1), it corresponds to the following property:

**Monotonicity in Connections (MC).** *A method  $H \in \mathcal{H}$  is monotonic in connections if for all  $(N, Q, G, x, r)$  and all  $i, j \in N : \{x_i = x_j \text{ and } f(i) \supseteq f(j)\} \Rightarrow y_i \geq y_j$ .*

Fix  $(G, x, r)$  and  $i, j$  as in the premises of MC with  $x_i = x_j = z$ . Let  $\varphi = H^{pro}(G, x, r)$ . Delete all resource-types except those in  $f(j)$ , and all agents except  $i, j$ . The reduced problem has the complete graph  $\{i, j\} \times f(j)$ , and claims  $x'_i = z - \varphi_{iQ \setminus f(j)}$ ,  $x'_j = z$ . Set  $\delta = \varphi_{iQ \setminus f(j)}$  and  $t$  to be the total resource available in the reduced problem. Note  $t \leq 2z - \delta$ . Consistency implies

$$y_j = \frac{z}{2z - \delta}t; \quad y_i = \frac{z - \delta}{2z - \delta}t + \delta \Rightarrow y_i \geq y_j$$

## 6 Extensions of uniform gains and uniform losses

A straightforward generalization of Problem (5) delivers a large family of edge-consistent bipartite methods.

**Lemma 1** *Fix a strictly convex function  $W$  and a convex function  $B$ , both from  $\mathbb{R}$  into itself. For any problem  $(N, Q, G, x, r) \in P$  the flow*

$$\tilde{\varphi} = \arg \min_{\varphi \in \mathcal{F}(G, x, r)} \sum_{ia \in G} W(\varphi_{ia}) + \sum_{i \in N} B(x_i - \varphi_{if(i)}) \quad (13)$$

*defines an edge-consistent, symmetric, and continuous bipartite rationing method.*

We explain in the next section that the typical method  $\tilde{H}$  defined by (13) does not meet RCG (see the Corollary to Lemma 2).

**Proof.** The flow  $\tilde{\varphi}$  is well defined because the objective function is strictly convex and finite. For Symmetry and Continuity we repeat the corresponding argument in the proof of Theorem 1. For Edge-CSY we fix  $(G, x, r)$ , an edge  $ia \in G$ , and let  $\tilde{\varphi}$  be given by (13). With the notation in the definition of Edge-CSY, observe that if  $\varphi' \in \mathcal{F}(G \setminus \{ia\}, x(-ia), r(-ia))$ , then adding  $ia$  to  $G$  and  $\tilde{\varphi}_{ia}$  to  $\varphi'$  yields a flow  $(\varphi', \tilde{\varphi}_{ia})$  in  $\mathcal{F}(G, x, r)$ . If the restriction of  $\tilde{\varphi}$  to  $P(-ia)$  is not optimal in that problem, we can then construct a flow  $(\varphi', \tilde{\varphi}_{ia})$  beating  $\tilde{\varphi}$  in  $P$ . In the reduced problem  $(G \setminus \{ia\}, x(-ia), r(-ia))$  the restriction  $\tilde{\varphi}^{-ia}$  of  $\tilde{\varphi}$  to  $G \setminus \{ia\}$  is clearly a max-flow and  $x(-ia) - \tilde{\varphi}_{if(i) \setminus \{a\}}^{-ia} = x_i - \varphi_{if(i)}$ , implying Edge-CSY. ■

The bipartite proportional method corresponds to  $W = B = En$ , and we know from Theorem 1, this is the only edge-consistent extension of the standard proportional method. By contrast, the bipartite rationing methods in Lemma 1 contain infinitely many extensions of uniform gains, and, in a limit sense, of uniform losses as well.

### 6.1 Extending uniform gains

It is well known (see De Frutos and Masso [20]) that for any  $(x, t) \in \mathcal{P}^0$ , and for any  $W$  strictly convex, we have  $h^{ug}(x, t) = \arg \min_{y \leq x, y_N = t} \sum_{i \in N} W(y_i)$ . Therefore setting  $B \equiv 0$  in (13) delivers a consistent bipartite extension  $H^W$  of  $h^{ug}$ , satisfying Merging of Identically Connected Resource-types.

**Proposition 1** *For any strictly convex function  $W$  from  $\mathbb{R}_+$  into itself, and any problem  $(N, Q, G, x, r) \in P$ , the flow*

$$\overset{\circ}{\varphi} = \arg \min_{\varphi \in \mathcal{F}(G, x, r)} \sum_{ia \in G} W(\varphi_{ia}) \quad (14)$$

*defines a method  $H^W$  in  $\mathcal{H}(\text{Edge-CSY}, \text{MIR})$  satisfying Merging of Identically Connected Resource-types, and extending the standard method  $h^{ug}$ . Different choices of  $W$  yield infinitely many different methods  $H^W$ .*

**Proof** We fix  $(G, x, r)$  and describe the Kuhn Tucker conditions characterizing  $\overset{\circ}{\varphi}$  in (14) with associated net shares  $y_i = \overset{\circ}{\varphi}_{if(i)}$ . If  $y_i < x_i$  for some agent  $i$ ,

then we must have  $\overset{\circ}{\varphi}_{ia} \geq \overset{\circ}{\varphi}_{ja}$  for all  $a \in f(i)$  and  $j \in g(a)$ : otherwise a transfer from  $\overset{\circ}{\varphi}_{ja}$  to  $\overset{\circ}{\varphi}_{ia}$  yields a better flow. Thus the KKT conditions: for all  $i, j$  and  $a \in f(i) \cap f(j)$

$$y_i < x_i \Rightarrow \overset{\circ}{\varphi}_{ia} = \max_{j \in g(a)} \overset{\circ}{\varphi}_{ja} \text{ for all } i, j \text{ and } a \in f(i) \cap f(j) \quad (15)$$

*Proof of the first statement.* By Lemma 1 we need only to check that  $H^W$  satisfies MIR (which implies RCG). But it is clear that if in  $(G, x, r)$  the types  $a, b$  have  $g(a) = g(b)$ , and the flow  $\varphi$  meets the KKT conditions (15), then merging the flows through  $a$  and  $b$  gives a new flow still meeting the KKT conditions, so we are done.

*Proof of the second statement.* If equal split of each resource-type  $a$  among  $g(a)$  is feasible (does not exceeds any claim), it will be  $\overset{\circ}{\varphi}$  for any  $W$ . When  $G$  is the complete graph, by RCG the profile of net shares is  $y = h^{ug}(x, r_Q)$ . However, even in the case  $G = N \times Q$ , the choice of  $W$  will matter because it may affect the optimal flow  $\overset{\circ}{\varphi}$ . Assume for instance  $N = \{1, 2\}$ ,  $x = (1, 4)$ ,  $Q = \{a, b\}$ ,  $r = (1, 3)$ . Then  $h^{ug}(x, r_Q) = (1, 3)$ , and the corresponding max-flows take the form

$$\varphi_{1a} = z; \varphi_{1b} = 1 - z; \varphi_{2a} = 1 - z; \varphi_{2b} = 2 + z$$

for some  $z \in [0, 1]$ . Choose  $W^1(z) = -z^2$ , and  $W^2(z) = \ln(z)$ , so that  $W^2$  guarantees  $\varphi_{ia} > 0$  for all  $ia \in G$  whereas  $W^1$  does not. Check that  $\arg \max_z \{W^1(z) + 2W^1(1 - z) + W^1(2 + z)\} = \{0\}$ , that is the single unit of type  $a$  goes to agent 2, who also gets 2 units of type  $b$ . On the other hand the optimal  $z$  for  $W^2$  is  $\frac{1}{2}(\sqrt{3} - 1)$ , so agent 2 gets 0.63 units of type  $a$  and 2.37 units of type  $b$ .

If  $G$  is not complete, even the shares  $y^1, y^2$  may differ. For an example we modify our earlier numerical example in Figure 2 by keeping the same graph  $G$ , but with claims  $x = (1, 1, 4)$  and resources  $r = (1, 4)$ . For any max-flow we have  $\varphi_{2b} < \varphi_{3b}$ , therefore (15) implies  $\varphi_{2a} + \varphi_{2b} = 1$  for any choice of  $W$ . The max-flows take the form

$$\varphi_{1a} = z; \varphi_{2a} = 1 - z; \varphi_{2b} = z; \varphi_{3b} = 4 - z$$

so for the same functions  $W^1, W^2$  we get  $z^1 = 0$  and  $z^2 > 0$ .

It is now clear that (14) defines infinitely many different bipartite methods. ■

## 6.2 Extending uniform losses

The standard uniform losses method obtains as  $h^{ul}(x, t) = \arg \min_{y \leq x, y_N = t} \sum_{i \in N} B(x_i - y_i)$ , for any  $B$  strictly convex (see again [20]). However setting  $W \equiv 0$  in (13) and choosing  $B$  strictly convex does not define a bipartite rationing method because it does not specify the entire flow  $\varphi$ , only the net shares  $y_i = \varphi_{if(i)}$ . But a lexicographic minimization, first of  $\sum_{i \in N} B(x_i - y_i)$  delivering the net shares  $\hat{y}$ , then of  $\sum_{ia \in G} W(\varphi_{ia})$  over

$\mathcal{F}(G, \dot{y}, r)$  does the trick. Note that the resulting flow is also the limit of the solution of (13) for the pair  $W, \mu B$  when the parameter  $\mu$  goes to infinity.

In the following we write the set of feasible net shares at problem  $(G, x, r) \in \mathcal{P}$  as

$$\mathcal{Y}(G, x, r) = \{y \in \mathbb{R}_{++}^N \mid \text{for some } \varphi \in \mathcal{F}(G, x, r) : y_i = \varphi_{if(i)} \text{ for all } i\}$$

**Proposition 2** *Fix any two strictly convex function  $W, B$  from  $\mathbb{R}_+$  into itself. For any problem  $(G, x, r) \in \mathcal{P}$ , the net shares*

$$\dot{y} = \arg \min_{y \in \mathcal{Y}(G, x, r)} \sum_{i \in N} B(x_i - y_i)$$

and the flow

$$\dot{\varphi} = \arg \min_{\varphi \in \mathcal{F}(G, \dot{y}, r)} \sum_{ia \in G} W(\varphi_{ia})$$

define a method  $H^{B \succ W}$  in  $\mathcal{H}(\text{Edge} - \text{CSY}, \text{MIR})$ , extending the standard method  $h^{ul}$ . The choice of  $B$  does not matter, but different choices of  $W$  yield infinitely many different methods  $H^{B \succ W}$ .

**Proof** The resulting flow is well defined because  $B, W$  are both strictly convex, and we already noted that it gives the uniform losses shares when  $|Q| = 1$ . The method is clearly symmetric, and Continuity follows from applying Berge's Maximum Theorem twice, once to  $(x, r) \rightarrow \dot{y}$ , then to  $(\dot{y}, r) \rightarrow \dot{\varphi}$ .

For Edge-CSY, we fix  $(G, x, r) \in \mathcal{P}$  and pick  $ia \in G$  and  $y' \in \mathcal{Y}(G \setminus \{ia\}, x(-ia), r(-ia))$ : the profile  $y : y_i = y'_i + \dot{\varphi}_{ia}, y_j = y'_j$  for  $j \neq i$ , is in  $\mathcal{Y}(G, x, r)$ . Moreover for any  $B$  we have  $\sum_{j \in N} B(x_j(-ia) - y'_j) = \sum_{j \in N} B(x_j - y_j)$ . Therefore the optimal net shares in the reduced problem are  $y' : y'_i = \dot{y}_i - \dot{\varphi}_{ia}, y'_j = y_j$  for  $j \neq i$ . Finally the separability of the objective function  $\sum_{ia \in G} W(\varphi_{ia})$  implies  $\varphi'_e = \dot{\varphi}_e$  for all  $e \in G \setminus \{ia\}$ .

For MIR, fix  $(G, x, r)$  and two types  $a, b$  such that  $g(a) = g(b)$ . Merging the flows through  $a$  and  $b$  leaves  $\mathcal{Y}(G, x, r)$  unchanged, hence  $\dot{y}$  as well; the flow  $\dot{\varphi} \in \mathcal{Y}(G, \dot{y}, r)$  is also merged as desired, by exactly the same argument as in the proof of Proposition 1.

We check next that the choice of  $B$  does not matter. This follows from the observation that we can represent  $\mathcal{Y}(G, x, r)$  as the core of a submodular cooperative game<sup>12</sup> in  $N$ , and the familiar fact that such a core has a Lorenz dominant element ([18]). Thus  $\{x\} - \mathcal{Y}(G, x, r)$  has a Lorenz dominant element as well, and a characteristic property of this vector is that for any strictly convex  $B$ , it minimizes  $\sum_{i \in N} B(z_i)$  over  $\{x\} - \mathcal{Y}(G, x, r)$ .

For the final statement about the infinite number of bipartite extension, we repeat the argument in the proof of Proposition 1. ■

<sup>12</sup>Setting the value of coalition  $S \subseteq N$  as  $v(S) = \min_{\emptyset \subseteq T \subseteq S} \{x_T + r_{f(S \setminus T)}\}$ , then  $y \in \mathcal{Y}(G, x, r) \Leftrightarrow y_S \leq v(S)$  for all  $S$ , with equality for  $S = N$  (see [9]). Then  $\{x\} - \mathcal{Y}(G, x, r)$  is the core of the *supermodular* game  $w(S) = x_S - v(S)$ .

## 7 Standard methods not consistently extendable

After establishing that the three benchmark methods are extendable to  $\mathcal{H}(\text{Node-}CSY)$ , it is natural to ask whether any consistent standard method  $h$  is extendable as well. The answer is no, because the combination of Node-CSY and RCG imposes the following necessary condition for extendability.

**Lemma 2** *Assume the set  $\mathcal{Q}$  of potential resource types is infinite, and pick any bipartite rationing method  $H \in \mathcal{H}(\text{Node} - CSY)$  with corresponding standard method  $h \in \mathcal{H}^0$ . Then for all  $(N, x, t) \in \mathcal{P}^0$  and all  $\delta \in [0, 1]$ , we have*

$$h(x - \delta \cdot h(x, t), (1 - \delta)t) = (1 - \delta) \cdot h(x, t) \quad (16)$$

**Proof** Fix  $(N, x, t) \in \mathcal{P}^0$ , two integers  $p, q, 1 \leq p < q$ , and a set  $Q$  of types with cardinality  $q$ . Consider the problem  $P = (N \times Q, x, r)$  where  $r_a = \frac{t}{q}$  for all  $a \in Q$ , with associated profile of shares  $y$  at  $\varphi = H(P)$ . By RCG  $y = h(x, t)$  and by symmetry  $\varphi_{ia} = \frac{y_i}{q}$  for all  $i \in N$ . Drop now  $p$  of the nodes and let  $Q'$  be the remaining set of types. Applying Node-CSY  $p$  successive times gives

$$H(N \times Q', x', r') = \varphi'$$

where  $x' = x - \frac{p}{q}y$ ,  $r'_a = \frac{t}{q}$  for all  $a \in Q'$ , and  $\varphi'$  is the restriction of  $\varphi$  to  $N \times Q'$ . Therefore  $y' = \frac{q-p}{q}y$ . RCG in the reduced problem gives  $y' = h(x', \frac{q-p}{q}t)$ . We just showed  $\frac{q-p}{q}y = h(x - \frac{p}{q}y, \frac{q-p}{q}t)$ , precisely (16) for  $\delta = \frac{p}{q}$ . Then continuity implies (16) for other real values of  $\delta$ . ■

Property (16) is a new axiom<sup>13</sup> in the theory of standard rationing methods. We distribute first a fraction  $\delta$  of the shares  $h(x, t)$ , and decrease accordingly individual claims before distributing the remaining  $(1 - \delta)t$  units of resource: the result is the same as if we distributed all  $t$  units in one shot.

It is easy to check directly that our three benchmark standard methods meet (16), which we already know because we showed in the previous sections they are consistently extendable. On the other hand many (if not most) standard methods discussed in the literature (see surveys [39],[31]) fail this property. We illustrate this fact first with two well known examples, the Talmudic method ([2]) and the family of equal sacrifice methods ([43], [31]), then with the methods defined in Lemma 1.

The Talmudic method  $h^T$  is a mixture of uniform gains and uniform losses in the following sense:

$$h^T(x, t) = h^{ug}\left(\frac{x}{2}, t\right) \text{ if } t \leq \frac{x_N}{2}; = \frac{x}{2} + h^{ul}\left(\frac{x}{2}, t - \frac{x_N}{2}\right) \text{ if } \frac{x_N}{2} \leq t \leq x_N$$

An *equal sacrifice* method is determined by a function  $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \cup \{-\infty\}$  with strictly positive derivative. The solution  $y = h^u(x, t)$  of  $(N, x, t) \in \mathcal{P}^0$  is defined by budget balance and

$$\text{for all } i: y_i > 0 \Rightarrow u(x_i) - u(y_i) = \max_N \{u(x_j) - u(y_j)\}$$

<sup>13</sup>It is reminiscent of the star-shaped invariance axiom in axiomatic bargaining theory: see the survey [41]. See also the discussion in [33].

The proportional method corresponds to  $u(z) = \ln(z)$ , and uniform losses to  $u(z) = z$ . Uniform gains is not an equal sacrifice method.

**Corollary 1** *The Talmudic method, and all equal sacrifice methods, except the proportional and uniform losses, are not extendable to  $\mathcal{H}(\text{Node} - \text{CSY})$ .*

**Proof.** For the Talmudic  $h^T$ , take  $n = 2$  and check that  $h^T((4, 2), 3) = (2, 1)$ , while  $h^T((4, 2) - \frac{1}{2}(2, 1), \frac{3}{2}) = (\frac{3}{4}, \frac{3}{4}) \neq \frac{1}{2}(2, 1)$ .

Fix now an equal sacrifice method satisfying (16), with corresponding benchmark function  $u$ . We must show that  $u$  is, up to a positive affine transformation,  $u(z) = \ln(z)$  or  $u(z) = z$ . Fix  $y_1, y_2, \varepsilon_1, \varepsilon_2$ , all positive, and such that

$$u(y_1 + \varepsilon_1) - u(y_1) = u(y_2 + \varepsilon_2) - u(y_2) \quad (17)$$

Then  $y = h(x, t)$  for  $x = y + \varepsilon$  and  $t = y_1 + y_2$ . Applying now (16) for  $\delta \in [0, 1]$ , we get

$$u(\delta' y_1 + \varepsilon_1) - u(\delta' y_1) = u(\delta' y_2 + \varepsilon_2) - u(\delta' y_2) \quad (18)$$

(recall the notation  $\delta' = 1 - \delta$ ). For fixed  $y$ , equation (17) defines on some positive interval  $[0, \alpha[$  a one-to-one function  $\varepsilon_1 \rightarrow \varepsilon_2$  with derivative at zero  $\frac{u'(y_2)}{u'(y_1)}$ . Equation (18) defines the *same* function for any  $\delta' \in [0, 1]$ , and its derivative at zero is now  $\frac{u'(\delta' y_2)}{u'(\delta' y_1)}$ , therefore

$$\frac{u'(y_2)}{u'(y_1)} = \frac{u'(\delta' y_2)}{u'(\delta' y_1)} \text{ for all } y_1, y_2 > 0 \text{ and all } \delta' \in [0, 1] \quad (19)$$

An affine transformation of  $u$  gives the same equal sacrifice method. So we can rescale  $u$  so that  $u'(1) = 1$  and take  $y_2 = 1$  in (19). We get

$$u'(ab) = u'(a)u'(b) \text{ for all } a, b \text{ s.t. } \min\{a, b\} \leq 1$$

This implies  $u'(a)u'(\frac{1}{a}) = 1$ . If  $\min\{a, b\} > 1$  we write  $u'((ab)\frac{1}{b}) = u'(ab)u'(\frac{1}{b})$  and deduce that  $u'(ab) = u'(a)u'(b)$  holds for all  $a, b > 0$ . This classic functional equation implies that  $u'$  is a power function, thus after another rescaling, the only possibilities are  $u(z) = z^p$  for  $p > 0$ , or  $u(z) = -z^p$  for  $p < 0$ , or  $u(z) = \log(z)$ . The latter is the proportional method, for which (16) is true. Ditto for the uniform losses method, corresponding to  $u(z) = z$ . But for any other method, (16) fails to be true. A simple way to check this is to fix  $y_1 = 2, y_2 = 4$ , choose  $a, b > 0$  such that the power  $p$  equal sacrifice method selects  $y$  in the problem  $x = (a + 2, b + 4)$ ,  $t = 6$ , and apply (16) for  $\delta = \frac{1}{2}$ . This writes as follows, for all positive  $a, b$ :

$$\{(a + 2)^p - 2^p = (b + 4)^p - 4^p\} \Rightarrow (a + 1)^p - 1 = (b + 2)^p - 2^p$$

Then one checks that the two curves defined respectively by the left equation and the right equation are distinct if  $p \neq 0, 1$ . ■

We turn to the Edge-Consistent methods  $H^{W,B}$  identified in Lemma 1. The corresponding standard method  $h^{W,B}$  computes the shares  $\tilde{y} = h^{W,B}(x, t)$  as follows:

$$\tilde{y} = \arg \min_{y \leq x, y_N = t} \sum_{i \in N} W(y_i) + \sum_{i \in N} B(x_i - y_i) \quad (20)$$

**Corollary 2** Assume  $W, B$  are both strictly convex. The standard method (20) meets (16) if and only if, up to normalization,  $W$  is the entropy function  $W^*(z) = z \ln(z)$ .

**Proof** We give the proof when  $W, B$  are both smooth; it extends to general strictly convex functions by a straightforward limit argument that we omit for brevity. We write  $w, b$  for the derivatives of  $W, B$ .

*Statement if.* Pick any  $(x, t) \in \mathcal{P}^0$  and set  $y = h^{En, B}(x, t)$ . From  $w(0) = -\infty$  follows that  $y_i > 0$  for all  $i$  s.t.  $x_i > 0$ , and the KKT conditions characterizing  $y$  are

$$\text{for all } i: y_i < x_i \Rightarrow \ln(y_i) + b(x_i - y_i) = \max_{j \in N} \{\ln(y_j) + b(x_j - y_j)\}$$

Now for  $\delta \in ]0, 1[$  and any  $i$  we have  $y_i < x_i \Leftrightarrow \delta' y_i < x_i - \delta y_i$ , where we use the notation  $\delta' = 1 - \delta$ . The above equality can be rewritten as

$$\ln(\delta' y_i) + b((x_i - \delta y_i) - \delta' y_i) = \max_{j \in N} \{\ln(\delta' y_j) + b((x_j - \delta y_j) - \delta' y_j)\}$$

so  $\delta' y$  meets the KKT conditions for  $h^{En, B}(x - \delta y, \delta' t)$ .

*Statement only if.* Because  $b$  increases strictly, for any  $y_1, y_2$ , positive and close enough, there exists  $z_1, z_2$ , positive and such that

$$w(y_1) - b(z_1) = w(y_2) - b(z_2)$$

Thus the shares  $y = (y_1, y_2)$  are an interior solution of the program (20) for the problem  $(z + y, t = y_1 + y_2)$ , i.e.,  $h^{W, B}(z + y, t) = y$ . Property (16) gives  $h^{W, B}(z + \delta y, \delta t) = \delta y$  for all  $\delta \in [0, 1]$ . Note that for  $\delta > 0$ ,  $\delta y$  is an interior solution of (20) for the problem  $(z + \delta y, \delta t)$ , therefore

$$w(\delta y_1) - b(z_1) = w(\delta y_2) - b(z_2) \text{ for all } \delta \in ]0, 1]$$

The two equations above give

$$w(y_1) - w(\delta y_1) = w(y_2) - w(\delta y_2)$$

for all  $y_1, y_2$  positive and close enough, and all  $\delta$  close to 1. Letting  $\delta$  go to 1, we get  $y_1 w'(y_1) = y_2 w'(y_2)$ , and we see that  $y_1 w'(y_1)$  is a positive constant. Thus  $w(z) = A w^*(z) + B$ , and  $W$  can be normalized to  $W^*$  without changing the standard method. ■

In the companion paper [33] we show that the set of standard methods satisfying (16) reduces essentially to the family  $h^{W^*, B}$ , dubbed loss calibrated methods; moreover all such methods are uniquely extendable to  $\mathcal{H}(\text{Edge} - \text{CSY}, \text{MIR})$  (the qualification refers to limit points of the family such as uniform gains and uniform losses).

## 8 Concluding Remarks

All symmetric *standard* rationing methods discussed in the literature, including the three benchmark methods, meet several natural monotonicity properties. The allocation of an agent is a weakly increasing function of the total amount of resource available, and of his own claim; it is weakly decreasing in other agent's claims. Finally, agents with larger claims receive larger allocations, and incur larger losses. We have not been able to prove analogs of these properties in the bipartite setting, not even for the proportional method. We conjecture that the bipartite proportional method satisfies these monotonicity conditions. Progress on this question may help us better understand questions associated with strategic aspects of the problem where agent claims may not be observable.

As mentioned just before Lemma 2, property (16) allowing us to dismiss large sets of standard methods in the two Corollaries, depends not only upon Node-CSY but also on RCG. Although we find the latter compelling, it is nevertheless interesting to understand which consistent standard methods extend to the bipartite framework as continuous, symmetric and node (or edge) consistent methods. We offer no conjecture toward answering this difficult question.

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## Appendix: Canonical Decomposition

We show that any overdemanded problem  $P \in \mathcal{P}$  can be uniquely decomposed into irreducible problems over a partition of agents and resources. Then we explain how a method defined only for irreducible problems is canonically extended into a full-fledged bipartite method. This construction is used in the proof of both Theorem 1, where several properties are proven first for irreducible problems, then extended to all overdemanded problems by means of the decomposition.

**Lemma 3** *For any problem  $P = (N, Q, G, x, r) \in \mathcal{P}$ , there is an integer  $K \geq 1$  and two partitions,  $N = \cup_1^K N^k$ ,  $Q = \cup_1^K Q^k$ , such that:*

- $g(Q^1) = N^1; \dots; g(Q^k) \setminus \{N^1 \cup \dots \cup N^{k-1}\} = N^k$  for all  $k, 2 \leq k \leq K$ ;
- for all  $k, 1 \leq k \leq K$ , the problem  $P^k = (N^k, Q^k, G(N^k \times Q^k), x_{[N^k]}, r_{[Q^k]})$  is irreducible; and if  $K > 1$ , the problems  $P^k$  for  $1 \leq k \leq K - 1$  are balanced.
- a flow  $\varphi$  in problem  $P$  is a max-flow if and only if it is the “union” of  $K$  max-flows  $\varphi^k$ , one in each subproblem  $P^k$ .

**Proof sketch** If  $P$  is irreducible only the coarsest partition can fit the bill. This is the only case where  $K = 1$ . If  $P$  is not irreducible, there is at least one “balanced” subset  $B$  of  $Q$ , i.e.,  $r_B = x_{g(B)}$ . Any two balanced subsets either are disjoint or their intersection satisfies the same property. Thus the inclusion minimal balanced subsets are disjoint, and they are the first elements  $Q^1, \dots, Q^k$ , of the partition of  $Q$ . The inductive construction continues on the problem reduced to  $N \setminus g(Q^1 \cup \dots \cup Q^k)$  and  $Q \setminus \{Q^1 \cup \dots \cup Q^k\}$ . ■

The canonical partition is unique up to possibly relabeling the  $P^k$ : if the first step delivers several inclusion minimal balanced subsets, we can exchange them freely; similarly if  $g(Q^k) \cap N^{k-1} = \emptyset$ , we can exchange  $P^k$  and  $P^{k-1}$ .

Extending a bipartite rationing method from the set  $\mathcal{P}^{ir}$  of irreducible problems to  $\mathcal{P}$  is done in the following way.

**Definition 5** *Given a method  $H^{ir}$  on  $\mathcal{P}^{ir}$ , its canonical extension  $H$  to  $\mathcal{P}$  selects for every  $P \in \mathcal{P}$  the max-flow  $H(P) = \varphi$  that is the union of the max-flows  $H^{ir}(P^k)$  for the decomposition above.*

Clearly the canonical extension of a method from  $\mathcal{P}^{ir}$  to  $\mathcal{P}$  does not depend on the labeling of the irreducible subproblems  $P^k$  of a given problem  $P$ .

**Definition 6** *The method  $H^{ir}$  on  $\mathcal{P}^{ir}$  is node consistent (resp. edge consistent) iff its canonical extension is.*

We cannot define consistency directly for methods on  $\mathcal{P}^{ir}$ , because the reduced problem of an irreducible one may not be irreducible. However for any method  $H^{ir}$  on  $\mathcal{P}^{ir}$ , its canonical extension is the only possible method on  $\mathcal{P}$  that extends  $H^{ir}$  and is node/edge consistent, so this is the right definition. In particular if the method  $H$  on  $\mathcal{P}$  is node-consistent, it is the canonical extension of its projection on  $\mathcal{P}^{ir}$ .

Note further that the canonical extension preserves symmetry, and if the method on  $\mathcal{P}^{ir}$  meets IMT or IFM, so does its canonical extension. But continuity is not guaranteed, as it requires some conditions linking the solutions for irreducible problems of different sizes.